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Abstract

Regular maps are cellular embeddings of graphs on surfaces with a transitive group of automorphisms on the triangles of the barycentric subdivision of the map. If these maps are both self-dual and self-Petrie-dual, while having all possible exponents, we call them super-symmetric. S. E. Wilson in his dissertation stated that a certain class of regular maps M_n for $n \geq 1$ are both self-dual and self-Petrie-dual, which he computationally verified for $n \leq 100$. Later on, D. Archdeacon, M. Conder and J. Širáň proved duality and self-duality of Wilson's maps for every $n \geq 1$ and they also proved their super-symmetry. Then, G. A. Jones proposed that Wilson's maps should be isomorphic to a parallel product of certain maps constructed from dihedral groups, and at the same time, they should also arise from triangle groups by factoring them with certain special normal subgroups derived from second-order commutator subgroups of triangle groups. In this dissertation we analyze both proposals of Jones and prove that the construction of Wilson's maps using parallel products of maps derived from dihedral groups with an additional automorphism of order 2 and the construction using the factorization of triangle groups are equivalent only for odd $n \geq 3$. We also extended our analysis to generalised dihedral groups and their parallel products.

Abstrakt

Regulárne mapy sú bunečné vnorenia grafov do plôch s tranzitívnou grupou automorfizmov na trojuholníkoch barycentrickej subdivízie mapy. V prípade, že sú tieto mapy samoduálne aj samo-Petrie-duálne, pričom majú všetky možné exponenty, nazývame ich super-symetrické. S. E. Wilson vo svojej PhD dizertácii navrhol skúmať istú triedu regulárnych máp ,pričom overil, že pre $n \leq 100$ sú tieto mapy samoduálne a aj samo-Petrie-duálne. Neskôr D. Archdeacon, M. Conder a J. Širáň to ukázali pre všetky prirodzené n a navyše dokázali, že dokonca všetky ich prípustné rotačné mocniny sú navzájom izomorfné, a teda, že ide o super-symetrické mapy. G. A. Jones potom uviedol, že Wilsonove mapy by mali byť paralelným produktom istých máp skonštruovaných z dihedrálnych grúp a zároveň by rovnako mali vzniknúť z trojuholníkových grúp ich faktorizáciou istými špeciálnymi normálnymi podgrupami. V našej dizertačnej práci analyzujeme obe Jonesove konštrukcie a dokazujeme, že konštrukcia Wilsonových máp pomocou paralelných produktov máp odvodených z dihedrálnych grúp s dodatočným automorfizmom rádu 2 a konštrukcia pomocou faktorizácie trojuholníkových grúp sú ekvivalentné len pre nepárne $n \geq 3$. Taktiež analyzujeme zovšeobecnené dihedrálne grupy a ich paralelné produkty.

Contents

1 Background

A map is a cellular embedding of a connected graph on a surface, or, equivalently, a cellular decomposition of a connected surface. A map automorphism, or, colloquially, a symmetry of a map, is a permutation of its flags (triangles of its barycentric subdivision) that preserves flag adjacencies. The group $Aut(M)$ of all automorphisms of a map M (under composition of permutations) acts semi-regularly on the flag set of M; if this action is regular the map M itself is said to be *regular*.

Regularity of a map M also implies transitivity of the induced action of $\mathrm{Aut}(M)$ on faces, vertices, edges and arcs (edges with direction). In particular, the length of each face boundary walk is the same and the valency of each vertex is constant; for future reference we will denote these quantities by l and m and speak about a regular map of *type* $\{l, m\}$.

Let z be a fixed flag of a regular map M. By regularity, reflections in the three sides of z are automorphisms of M, which are usually denoted r_i for $i \in \{0, 1, 2\}$, the subscript referring to i-dimensional objects (vertex, edge, and face incident to z) being moved by r_i (the remaining two objects being preserved by r_i). The compositions r_0r_2, r_1r_2 and r_0r_1 then are, respectively, automorphisms which locally rotate the map about the centre of an edge, about a vertex incident to the edge, and about a face incident to both. If M has type $\{l, m\}$, connectivity implies that the group $G = Aut(M)$ admits a presentation of the form

$$G = \mathrm{Aut}(M) = \langle r_0, r_1, r_2 \mid r_0^2, r_1^2, r_2^2, (r_0r_2)^2, (r_1r_2)^m, (r_0r_1)^l, ...\rangle,$$

where dots indicate relators needed to complete the presentation. Regularity further implies that the flag set of M can be identified with the set $\{g(z) \mid g \in G$, and hence with the group G acting on itself by left multiplication.

Without going into too much detail and mimicking the way we introduced regular maps, one may check that for a map M on an orientable surface, an automorphism of M that preserves the orientation of the supporting surface of M (in short, an orientation-preserving automorphism) is completely determined by its image of any particular arc. It follows that the group $\mathrm{Aut}^+(M)$ of all orientation-preserving automorphisms of M acts semi-regularly on the arcs (not flags) of M. If this action is transitive, and hence regular, we speak about an orientably-regular map, or a rotary map in the terminology of [14]. Such maps correspond to those which intuitively exhibit the 'highest level of orientation-preserving symmetry'.

Regular maps of type $\{\ell, m\}$ on simply connected surfaces (a sphere, a Euclidean plane, or a hyperbolic plane, according to whether $1/\ell + 1/m$ is greater than, equal to, or smaller than $1/2$) are of special importance ane are known as *universal tessellations*

$U(\ell, m)$. Their orientation-preserving automorphism groups are known as *(ordinary) triangle groups* $\Delta^+(\ell, m)$ and can be presented in the form

$$\Delta^+(\ell, m) = \langle R, S \mid R^m, S^2, (RS)^\ell \rangle$$

Every orientably-regular map M of type $\{\ell, m\}$ is then a smooth normal quotient of $U(\ell, m)$, and, similarly, the group $Aut^+(M)$ is a smooth normal quotient of $\Delta^+(\ell, m)$, with presentation

$$\mathrm{Aut}^+(M) = \langle r, s \mid r^m, s^2, (rs)^l, ... \rangle$$

An orientably-regular map M given by a presentation of its group $H = \mathrm{Aut}^+(M)$ will be denoted $M = (H; r, s)$.

The universal maps $U(\ell, m)$ are actually regular and their full automorphism group (including orientation-reversing symmetries) is known as the *full* or *extended* triangle group $\Delta(\ell, m)$, with presentation

$$\Delta(\ell, m) = \langle R_0, R_1, R_2 \mid R_0^2, R_1^2, c^2, (R_0R_1)^\ell, (R_1R_2)^m, (R_2R_0)^2$$

A regular map M with $Aut(M) = \langle r_o, r_1, r_2 \rangle$ as before is again a smooth normal quotient of $U(\ell, m)$, and the group $Aut(M)$ is a smooth normal quotient of $\Delta(\ell, m)$. For such a regular map M with $G = Aut(M)$ we will use the notation $M = (G; r_0, r_1, r_2)$, or $M = (G; a, b, c)$ if it is desirable to avoid subscripts.

Let $(G; a, b, c)$ and $(G'; a', b', c')$ be two regular maps. The two maps are known to be isomorphic if and only if there exists an isomorphism ϕ where: $G \to G'; \phi(a) = a',\ \phi(b) = b',\ \phi(c) = c'$.

New maps can be constructed from old ones with the help of map operators, and we will consider in detail the operators of duality, Petrie duality and rotational powers. We will only consider regular maps in what follows, although the three types of operators are applicable to general maps.

Let $M = (G; r_0, r_1, r_2)$ be a regular map. The operator of duality assigns the *dual map* $D(M)$ to M, defined by $D(M) = (G; r_2, r_1, r_0)$, that is, by swapping the roles of the involutions r_0 and r_2. Loosely speaking, the dual can be obtained from the given map by interchanging its vertices with faces and its faces with vertices. If the map M is isomorphic to the map $D(M)$, the map is called *self-dual*. Let us point out that the operator of duality preserves the supporting surface of the map M.

Another operator we will introduce is the one of Petrie-duality, assigning to a regular map $M = (G; r_0, r_1, r_2)$ its *Petrie dual* $P(M) = (G; r_0 r_1, r_1, r_2)$. The Petrie dual arises as a different embedding of the underlying graph of M, with new faces determined as follows: if we select $r_0 r_1$ as the first operator instead of r_0, face boundary walks will be created by walking along edges, switching sides in each midpoint and following all corners. These movements create closed walks which are called Petrie walks (also known as zigzag walks because of their 'zigzag' shape). All Petrie walks have the same length and each edge is included in exactly two distinct walks, therefore Petrie walks can be considered as faces of the Petrie-dual map $D(M)$ which are called *Petrie polygons.*

The operator of Petrie-duality preserves the underlying graph but does not preserve supporting surface in general. In case there exists an isomorphism between maps M and $P(M)$, the map M is *self-Petrie-dual.*

Along with operators of duality and Petrie duality we have also so called *hole operators* or *rotational powers* that create new maps from old ones while keeping the underlying graph and the automorphism group unchanged.

To explain this, let $M = (G; r_0, r_1, r_2)$ be a regular map of valency m. For every j relatively prime to m, the j-th rotational power $M^{(j)}$ is defined as the regular map $(G; r_0, r_1', r_2)$. When the j-th rotational power $M^{(j)}$ is isomorphic to the original map M we say that j as an element of the group of units *mod m* is *an exponent* of M.

A regular map M of valency m will be called *super-symmetric* if it is self-dual, self-Petrie-dual and if every unit mod m is its exponent. The type of such a super-symmetric map is necessarily $\{m, m\}$.

Next, we introduce useful algebraic tools for constructing new regular maps from old ones. Let $M_1 = (G; x_1, y_1, z_1)$ and $M_2 = (G_2; x_2, y_2, z_2)$ be a pair of regular maps, where, for $i \in \{1, 2\}$, the groups

$$G_i = \langle x_i, y_i, z_i | x_i^2, y_i^2, z_i^2, (x_i z_i)^2, (y_i z_i)^{m_i}, (x_i y_i)^{\ell_i}, ...\rangle$$

are presented as in (1), so that M_i is of type $\{\ell_i, m_i\}$. (The canonical generators r_0, r_1, r_2 have been replaced here by x_i, y_i, z_i to avoid double subscripts.) The parallel product $M_1||M_2$ is the map $M = (G; x, y, z)$, where $G = G_1||G_2$ in the sense of Wilson is the parallel product of the groups G_1 and G_2, defined as the subgroup of $G_1 \times G_2$ generated by the pairs $x = (x_1, x_2)$ $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Observe that M has type $\{\ell, m\}$, where $\ell = \mathrm{ord}(xy) = \mathrm{lcm}(\ell_1, \ell_2)$ and $m = \mathrm{ord}(yz) = \mathrm{lcm}(m_1, m_2)$.

We now describe a different but equivalent way of introducing parallel profuct of maps, following Jones [7] and based on using the extended (and generalised) triangle group $\Gamma = \Delta(2, \infty, \infty)$, with presentation

$$\Gamma = \langle X, Y, Z | X^2, Y^2, Z^2, (XZ)^2\rangle, \tag{1}$$

isomorphic to a free product $\langle X, Z\rangle * \langle Y\rangle \simeq (C_2 \times C_2) * C_2$ of the Klein four-group (also often denoted V_4) with C_2. If, for $i \in \{1, 2\}$, $\theta_i : \Gamma \to G_i$ are the natural epimorphisms and $K_i = \ker(\theta_i)$, then $G_1||G_2 \simeq G/K$ with $K = K_1 \cap K_2$ and the parallel product $M = M_1||M_2$ can equivalently be described by $M = (G/K; XK, YK, ZK)$.

The construction of Jones can be extended to any finite number of maps in an obvious way and we omit details.

2 Research stimuli

S. E. Wilson in the course of preparation for his dissertation thesis [12] in 1976 suggested to study a family of regular maps with automorphism groups defined by a specific presentation. For every $n \geq 1$ let W_n be a group given by the presentation

$$\langle a, b, c | a^2, b^2, c^2, (ac)^2, (ab)^{2n}, (bc)^{2n}, (abc)^{2n}, [(cb)^2, (ba)^2] \rangle \quad (2)$$

It can be shown [14, 1] that W_n has order $8n^3$, thus giving rise to a family MW_n of finite regular maps

$$MW_n = (W_n; a, b, c)$$

of type $\{2n, 2n\}$; to adhere to the previous notation here we have used a, b, c for r_0, r_1, r_2, respectively. Wilson computationally verified that for every $n \leq 100$ the regular map MW_n is, in addition, both self-dual and self-Petrie-dual. Later, D. Archdeacon, M. Conder and J. Širáň proved that these maps are not only self-dual and self-Petrie-dual for all $n \geq 1$ but that they are invariant also to all admissible exponents. Therefore, they proved that the maps MW_n defined by (2) are super-symmetric. Note that the length of Petrie walks in MW_n, equal to the order of abc, is also $2n$.

Two interesting alternative constructions of the family MW_n of Wilson's super-symmetric maps were suggested by Jones [7], and in what follows we describe both.

The first is based on considering a particular normal subgroup of finite index in the group Γ given by (1). As usual, let Γ' be the commutator subgroup of Γ and let $(\Gamma')' = \Gamma''$ be the second-order commutator subgroup of Γ. Further, let $(\Gamma')^{(n)}$ be the subgroup of Γ' generated by all n-th powers of elements of Γ'. Both Γ'' and $(\Gamma')^{(n)}$ are characteristic subgroups of Γ, so that their product $L = \Gamma''(\Gamma')^{(n)}$ is a normal subgroup of Γ; because Γ is generated only by 3 involutions and because of the presence of commutators and n-th powers in L the quotient Γ/L is finite. In fact, one can see that commutators $(cb)^2$ and $(ba)^2$ are generators of the normalizer Γ' of the subgroup of Γ. Digging into the presentation a little deeper one can realize that except for a^2, b^2, c^2 and $(ac)^2$, the only relators included in the presentation (2) are $(cb)^2$ and $(ba)^2$ raised to the n-th power along with the n-th power of the product $(cb)^2(ba)^2 = (cba)^2$, and the commutator $[(cb)^2, (ba)^2]$.

It follows that Γ/L is indeed isomorphic to the Wilson group W_n introduced in (2).

The second suggestion of Jones made in [7] was that the super-symmetric maps MW_n for each $n \geq 1$ may be constructed as parallel products of regular maps derived from extended dihedral group D_{2n}^* obtained from the group

$$D_{2n} = \langle r, s | r^{2n}, s^2, (rs)^2 \rangle$$

by adjoining an automorphism of order 2 inverting both r and s.

In more detail, for even $n \geq 1$ the group D_{2n}^* turns out to be an epimorphic image of the parent group Γ of the form (1). Mimicking Jones's construction based on epimorphisms $\Gamma \to D_{2n}^*$ for a fixed $n \geq 1$ (and hence fixed target group $G = D_{2n}^*$), in [7] it is suggested that the map MW_n is isomorphic to a parallel product determined by the intersection of all distinct kernels of epimorphisms $\Gamma \to D_{2n}^*$.

The two suggestions of Jones [7] actually provided an impetus for this thesis and for the papers [4, 5]. Namely, it is not clear at all if the two descriptions result in isomorphic maps. In fact, and to our surprise, this turned out to be true only for odd values of n, and for even n the parallel product results in a map obtained from MW_n by a factorisation by a (normal) subgroup isomorphic to C_2. Loosely speaking, for even $n \geq 2$ the parallel product construction by Jones gives only *a half* of Wilson's map MW_n.

3 Results

For an arbitrary positive integer n, we will consider the dihedral group D_{2n} of order $4n$, with a cyclic subgroup of order $2n$, defined by the presentation

$$\Delta^+_{2n} \simeq D_{2n} = \langle r, s \mid r^{2n}, s^2, (rs)^2 \rangle \tag{3}$$

This group defines an orientably-regular map M of a dipole of valency n in a sphere, whereby r, s and rs represent rotations of the map about a fixed vertex, the mid point of a fixed edge incident with the vertex, and the midpoint of a face incident with both the vertex and the edge, respectively.

Algebraically, the group (3) can be extended by adjoining an involution, say, t, such that $trt = r^{-1}$ and $tst = s$ (that is, t and s commute). The three elements generate the *extended* (classical) dihedral group D^*_{2n} with presentation

$$D^*_{2n} = \langle r, s, t \mid r^{2n}, s^2, t^2, (rs)^2, (rt)^2, (st)^2 \rangle \ . \tag{4}$$

Letting $x = st$, $y = rt$ and $z = t$, the presentation (4) defines a regular map $M_{2n} = M(G; x, y, z) = M(st, rt, t)$. The valency of this map is equal to $2n$ and face length is 2. The map is then of type $\{2n, 2\}$, with automorphism group $G = D^*_{2n} = \langle x, y, z \rangle = \langle r, s, t \rangle$. The subgroup $\langle r, s \rangle \simeq D_{2n}$ given by (4) is the *rotation group* of the map, consisting of all its orientation-preserving automorphisms.

The dual map of M_{2n} is defined by $D(M_{2n}) = (G; t, rt, st)$; it has type $\{2, 2n\}$ and can be visualised as an equatorial cycle of length $2n$ on a sphere. The Petrie-dual map $P(M_{2n})$ is of the form

$P(M_{2n}) = (G; s, rt, t)$ and is of type $\{2n, 2n\}$. The rotation group (that is, the group of all orientation-preserving automorphisms) of the Petrie dual is generated by r and st and hence is Abelian, in contrast with the rotation group of the original map (which is dihedral and hence non-Abelian for $n \geq 2$).

Summing up, these three maps, M_{2n}, $D(M_{2n})$ and $P(M_{2n})$ are pairwise non-isomorphic. Application of operators of duality and Petrie-duality can be repeated but due to the fact that $\langle P, D\rangle \cong S_3$ one obtains only three more maps this way, with automorphism group presentations as follows:

$DP(M_{2n}) = (G; s, rt, st)$, $PD(M_{2n}) = (G; t, rt, s)$ and finally $DPD(M_{2n}) = PDP(M_{2n}) = (G; st, rt, s)$.

But among the six maps obtained by a repeated use of the operators D and P only three are pairwise non-isomorphic, because the automorphism of D^*_{2n} that fixes rt and interchanges s with t implies that $P(M_{2n}) \simeq DP(M_{2n})$, $D(M_{2n}) \simeq PD(M_{2n})$ and $M_{2n} \simeq DPD(M_{2n}) \simeq PDP(M_{2n})$.

Finally, we will look at rotational powers and possible exponents of the maps M_{2n}. By our earlier description of rotational powers, if j is an arbitrary unit $\pmod{2n}$, the j-th rotational power $M_{2n}^{(j)}$ is the regular map $M(st, r^j t, t)$. Since the assignment $r \mapsto r^j$, $s \mapsto s$, $t \mapsto t$ extends to an automorphism of the group D^*_{2n}, it follows that the regular maps M_{2n} and $M_{2n}^{(j)}$ are isomorphic, and every unit $\pmod{2n}$ is an exponent of M_{2n}.

We are now ready to present the main findings of our dissertation. In the process we will use the same numbering of results as in our Dissertation, together with page numbers.

To put the suggestions made by Jones in [7] under scrutiny, we first determined the intersection of kernels of all epimorphisms $\Gamma \to D^*_{2n}$, where the parent group Γ is the one of (1) but presented in the form

$$\Gamma = \langle R, S, T | S^2, T^2, (ST)^2, (RT)^2 \rangle \tag{5}$$

This was quite complicated and resulted in:

Proposition 1 [page 33]: *For $n > 1$ the number of epimorphisms $\Gamma \to D^*_{2n}$ is $96\, n\varphi(n)$ if n is even, and $72\, n\varphi(n)$ if n is odd.*

We also determined the order of the automorphism group of the extended (classical) dihedral grooup (D^*_{2n} given by (4) with te following outcome.

Proposition 2 [page 33]: For $n > 1$ the number of automorphisms of the group D^*_{2n} is $32\, n\varphi(n)$ if n is even, and $24\, n\varphi(n)$ if n is odd.

Propositions 1 and 2 have the following immediate consequence.

Corollary 1 [page 33]: *The epimorphisms $\Gamma \to D^*_{2n}$ have exactly three distinct kernels.*

Recall that the parallel product $M_{2n} \times P(M_{2n}) \times D(M_{2n})$

is defined by the subgroup H of $G \times G \times G$ for $G = D^*_{2n} = \langle x, y, z\rangle$, where H is the subgroup of $G \times G \times G$ generated by three triples of involutions (x, x_P, x_D), (y, y_P, y_D) and (z, z_P, z_D). An analysis carried out on pages 34 to 37 of our Dissertation, which is essentially equivalent to determination of Smith's normal form for the Abelian part of H, led to a surprising outcome: while the order of H turned out to be $8n^3$ for every odd $n > 1$ as expected, for even n we found that $|H| = 4n^2$, a half of the expected order. Working further up to deriving presentations for H, for odd $n > 1$ we arrived at Wilson's groups, that is, for odd $n > 1$ we found that

$$H = \langle a, b, c \mid a^2, b^2, c^2, (ac)^2, (ab)^{2n}, (bc)^{2n}, [(ab)^2, (bc)^2]\rangle \qquad (6)$$

For even n, however, the group H of order $4n$ turned out to be determined by the presentation generated by the same involutions a, b, c but with relators

$$\langle a^2, b^2, c^2, (ac)^2, (ab)^{2n}, (bc)^{2n}, [(ab)^2, (bc)^2], (a(bc)^n)^2\rangle \qquad (7)$$

All these findings are summarized in the following Theorem.

Theorem [page 37]: *Let $n > 1$ and let M be the fully regular orientable map arising from the parallel product $M_{2n} \times P(M_{2n}) \times D(M_{2n})$. If n is odd, the full automorphism group of M has order $8n^3$ and admits a presentation (6); this group is isomorphic to the Wilson group given by (2). If n is even, then the full* *automorphism group of M has order $4n^3$ and has a presentation of the form (7); the group is in this case a quotient of the Wilson group (2) by a normal subgroup of order 2. In both cases, M is super-symmetric.*

In Chapter 5 of our dissertation we extended our investigation of constructions of super-symmetric maps to generalized dihedral groups. These groups, denoted $G_{n,e}$ in Chapter 5, are defined by the presentation $D_{n,e} = \langle r, s \mid r^{2n}, s^2, srsr^{-e} \rangle$. By letting $x = st,\ y = rt,\ z = t$ in a presentation (4) we obtain the regular map $M_{n,e}$, with automorphism group $G = \mathrm{Aut}(M_{n,e})$ presented in the form $G = \langle x, y, z \mid x^2, y^2, z^2, (xz)^2, (zy)^m, (yx)^\ell, \ldots \rangle$, so that

$$M_{n,e} = (G; x, y, z) = (G; st, rt, t).$$

This map has type $\{\ell, m\} = \{\frac{4n}{\gcd(2n,e+1)}, 2n\}$. The dual $D(M_{n,e})$ and the Petrie dual $P(M_{n,e})$ of the map $M_{n,e}$ have the form $D(M_{n,e}) = (G; t, rt, st)$ and $P(M_{n,e}) = (G; s, rt, t)$ and types $\{2n, \frac{4n}{\gcd(2n,e+1)}\}$ and $\{\frac{4n}{\gcd(2n,e-1)}, 2n\}$ respectively. On pages 39 to 41 of our Dissertation we proved the following Lemma.

Lemma 1 [page 42]: *Let n be a positive integer divisible by 4 and let $e \neq \pm 1$ be a residue class $\pmod{2n}$ such that $e^2 \equiv 1 \pmod{2n}$. Then the following holds:*

(i) If $e \equiv n-1 \pmod{2n}$, then $M_{n,e} \simeq PDP(M_{n,e})$, $P(M_{n,e}) \simeq DP(M_{n,e})$ and $D(M_{n,e}) \simeq PD(M_{n,e})$.

(ii) If $e \equiv n+1 \pmod{2n}$*, then* $M_{n,e} \simeq D(M_{n,e})$*,* $P(M_{n,e}) \simeq PD(M_{n,e})$ *and* $DP(M_{n,e}) \simeq PDP(M_{n,e})$*.*

Because of their exceptionality we call the values $n \pm 1 \bmod 2n$ *median*. The maps $M_{n,e}$ admit all admissible exponents, which follows from the fact that the mapping $\varphi : r \mapsto r^j$, $s \mapsto s$, $t \mapsto t$ gives an automorphism of the group $D^*_{n,e}$.

We now proceed analogously as in the study of products based on classical dihedral groups, but with detail that are even more complex. The first task is to count the number of epimorphisms from the parent group Γ given by the presentation (5). Findings we gathered in this area can be stated as follows. Looking for all epimorphisms from the group

$$\Gamma = \langle R, S, T | S^2, T^2, (ST)^2, (RT)^2 \rangle$$

to the extended generalized dihedral group $D^*_{n,e}$ evolved to the following task; determine the ratio $|\mathrm{Epi}(\Gamma \to D^*_{n,e})| / |\mathrm{Aut}(D^*_{n,e})|$. Calculations carried out on pages 42 to 44 give the following main results:

Proposition 3 [page 43]: *Let* $n > 1$ *be a positive integer and let* e *be a residue class* $\pmod{2n}$ *such that* $e^2 \equiv 1 \pmod{2n}$ *and* $e \neq \pm 1$*. The number of distinct epimorphisms from the group* Γ *onto the group* $D^*_{n,e}$ *is equal to* $6\,\nu_e \varphi(2n)$*.*

Proposition 4 [page 44]: *Let* $n > 1$ *be a positive integer and let* $e \neq \pm 1$ *be a residue class* $\pmod{2n}$ *such that* $e^2 \equiv 1$

(mod $2n$). *The order of the automorphism group of* $D^*_{n,e}$ *is equal to*

$$|\mathrm{Aut}(D^*_{n,e})| = \begin{cases} \nu_e\,\varphi(2n) & \textit{if } e \textit{ is not median,} \\ 4n\,\varphi(2n) & \textit{if } e \textit{ is median.} \end{cases} \tag{8}$$

From this information we can make the following conclusion.

Conclusion [page 44]: *If* $e \neq \pm 1$*, then the epimorphisms* $\Gamma \to D^*_{n,e}$ *have exactly 3 distinct kernels if* e *is median, and exactly 6 distinct kernels if* e *is not median.*

The surprising outcome is that, up to two exceptions (the median values) the number of epimorphisms in the above Conclusion does not depend on e at all.

Although in general the above epimorphisms have 6 distinct kernels, we nevertheless show that it is sufficient to take only 3 of them to form super-symmetric maps from extended generalized dihedral groups. Let $\tilde{M} = \tilde{M}_{n,e}$ be the parallel product $M_{n,e} \times PD(M_{n,e}) \times DP(M_{n,e})$.

Calculations similar to these done in the case of classical dihedral groups but made harder because of the extra parameter e, and again based on the analysis of the largest Abelian subgroup of $Aut(\tilde{M}_{n,e})$ and replacing considerations of its Smith normal form with a more direct approach, give the following surprising result. First, the group $H = Aut(\tilde{M}_{n,e})$ does not depend on e at all, again! Second, the presentations of H for odd $n > 1$ and even $n \geq 2$, respectively, are exactly the same as in the classical case (and, reiterating, they do not depend on e). In particular, one obtains

Theorem 2 [page 46] : *Let $n > 1$ be an arbitrary integer and let $e \neq \pm 1$ such that $e^2 \equiv 1 \pmod{2n}$. Further, let $\tilde{M}$ be the fully regular orientable map arising from the parallel product $M_{n,e} \times PD(M_{n,e}) \times DP(M_{n,e})$. If n is odd, the full automorphism group of $\tilde{M}$ has order $8n^3$ and admits a presentation (6); this group is isomorphic to the Wilson group given by (2). If n is even, then the full automorphism group of $\tilde{M}$ has order $4n^3$ and has a presentation of the form (7); the group is in this case a quotient of the Wilson group (2) by a normal subgroup of order 2. In both cases $\tilde{M}$ is super-symmetric.*

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