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Dissertation Thesis Abstract

Numerical modeling of some flow problems

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Abstract

We present a compact implicit well-balanced high-resolution numerical scheme for conservation and balance laws. The numerical scheme is derived by the approximation of the dominant error term of the first order implicit numerical scheme. We use the finite Taylor series and the Lax-Wendroff procedure, where mixed space-time derivatives are used. For the systems of equations, we express the numerical flux functions using the characteristic variables. For the balance laws, we extend the numerical scheme to be well-balanced, and we preserve the stationary solutions for the Burgers' equation with source term and for the shallow water equations with topography. Several numerical experiments are performed to show the properties of the numerical scheme.

Abstrakt

V práci prezentujeme kompaktnú implicitnú dobre vyváženú numerickú schému vysokého rozlíšenia pre zákony zachovania a rovnováhy. Numerická schéma je odvodená pomocou aproximácie chybového člena metódy prvého rádu presnosti. Použijeme konečný Taylorov rozvoj a Lax-Wendroff procedúru, v ktorej sú použité zmiešané časové a priestorové derivácie. Pre systémi rovníc vyjadríme numericke funkcie tokov pomocou charakteristických premenných. Pre zákony o rovnováhe rozšírime numerickú schému aby bola dobre vyvážená a zachovávame stacionárne riešenia pre Burgersovú rovnicu so zdrojovým členom a pre rovnice plytkej vody s topografiou. Vykonali sme viacero numerických experimentov aby sme prezentovali vlastnosti numerickej schémy.

Contents

1 Introduction

The world around us is full of interesting natural phenomena, and thanks to human curiosity, we are able to understand some of them. In applied mathematics, people develop mathematical models for such natural phenomena. For many fluid flow processes, the hyperbolic partial differential equations [18] were derived.

Numerical methods are used to obtain the solutions of the hyperbolic PDEs. We aim to time discretization methods with mixed temporal and spatial discretization methods based on the Taylor series expansion and using the Lax-Wendroff procedure. For hyperbolic problems, this approach is used in the context of the neoconservative advection equation used in the level set methods in [8, 10], where the resulting algebraic system has a simpler form. This type of method was introduced in finite volume framework for the scalar advection method in [22, 23], and applied in [9, 14, 15]. In the parametric form and for the linear advection equation in the conservative form, the scheme was presented in [7]. For the neoconservative advection equation for level set methods, the scheme is presented in [11]. The compact implicit high-resolution scheme derived for the nonlinear conservation laws is presented in [12] and uses the essentially non-oscillatory (ENO) method [27, 28] or the high-resolution limiter [6, 12].

The numerical scheme is well-balanced if it is able to preserve the stationary solutions of the balance laws. Our aim is to propose well-balanced compact implicit schemes that can preserve exact stationary solutions, if available. The stationary solutions for the shallow water equations with zero velocity are called "Lake at rest", and with nonzero velocity are called "moving water equilibria". Various methods are developed to preserve all such stationary solutions described, for example, in [1, 3, 4, 5, 13, 16, 17, 21, 24, 25, 26, 29]. In the context of the compact implicit scheme, such a well-balanced property was introduced in [30].

The derivation of the scheme is described using the approximation of the dominant error term of the first order method. To approximate the error term, Taylor's expansion and the procedure similar to the Lax-Wendroff procedure [27] is used. Furthermore, the parametric finite difference approximations are used to obtain the second order accurate scheme. Using solution-dependent values of the scheme parameters computed by, e.g. ENO or high-resolution limiter, the high-resolution form of the scheme is achieved. For the systems of equations, we use the approach of the characteristic variables [2, 6, 19]. For the well-balancing, the compact implicit high-resolution scheme approach similar to [13, 26] is used to preserve the "Lake at rest" and "moving water" stationary solutions.

2 Scalar conservation laws

The scalar conservation laws can be represented by the non-linear hyperbolic equation in the following form

$$\partial_t q + \partial_x f(q) = 0\,, \quad q(x,0) = q^0(x)\,, \quad x \in \Omega \subset \mathcal{R}\,, \quad t > 0, \tag{1}$$

where x and t are the spatial and temporal variable, respectively, $q = q(x,t)$ represents the unknown function with initial values prescribed by a given function q^0 and $f = f(q)$ is a given smooth flux function.

The numerical scheme can be written in the standard form of conservative schemes [18] for the equation (1)

$$q_i^{n+1} + \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^{n+1} - F_{i-1/2}^{n+1}\right) = q_i^n\,, \tag{2}$$

where the numerical fluxes $F_{i\pm 1/2}^{n+1}$ are specified by the numerical flux functions.

To derive our numerical method, the approach of the fractional step method is used [20]. Firstly, one chose flux splitting, where the flux function f is split into the sum of two functions f^+ and f^- having nonnegative and nonpositive derivatives

$$f = f^+ + f^-\,, \qquad \frac{df^+}{dq} \geq 0\,, \quad \frac{df^-}{dq} \leq 0\,, \quad q \in \mathcal{R}\,.$$

There are several options for flux splitting, and one of our choices is the Lax-Friedrichs (L-F) flux vector splitting [28] defined as follows

$$f^+(q) = \frac{1}{2}\left(f(q) + \alpha q\right) \ , \quad f^-(q) = \frac{1}{2}\left(f(q) - \alpha q\right) \ ,$$

where the parameter α is fixed at the maximum value of $|f'(q)|$ over the considered values of q.

We replace the numerical scheme written in the standard form (2) using the fractional step method combined with the fast sweeping method [20]. The approach consists of two partial steps, where each partial scheme is solved algebraically in the downwind direction. The first step, the forward step, now has the form

$$q_i^{n+1/2} + \frac{\Delta t}{\Delta x} F_{i+1/2}^{+,n+1/2} = q_i^n + \frac{\Delta t}{\Delta x} F_{i-1/2}^{+,n+1/2} \ , \ i = 1, 2, \ldots, I \ . \quad (3)$$

The second step, the backward step, has the form

$$q_i^{n+1} - \frac{\Delta t}{\Delta x} F_{i-1/2}^{-,n+1} = q_i^{n+1/2} - \frac{\Delta t}{\Delta x} F_{i+1/2}^{-,n+1} \ , \ i = I-1, I-2, \ldots, 0 \ . \quad (4)$$

To obtain a complete numerical scheme, one has to suggest an appropriate numerical flux function $F_{i+1/2}^{+,n+1/2}$ for the forward step (3) and $F_{i-1/2}^{-,n+1}$ for the backward step (4) to approximate $f^+(q)$ and $f^-(q)$. If we use the approximation as in [20], we obtain the first order implicit scheme in the following form for the forward step

$$q_i^{n+1/2} + \frac{\Delta t}{\Delta x}\left(f^+(q_i^{n+1/2}) - f^+(q_{i-1}^{n+1/2})\right) = q_i^n \ ,$$

and for the backward step

$$q_i^{n+1} + \frac{\Delta t}{\Delta x}\left(f^-(q_{i+1}^{n+1}) - f^-(q_i^{n+1})\right) = q_i^{n+1/2}\,.$$

The advantage of the described first order implicit numerical scheme is its stability and that each algebraic equation contains only one unknown q_i^{n+1}, which is a consequence of the upwind principle. The main disadvantage is the low accuracy of the scheme. We aim to preserve the advantages of the scheme and improve accuracy.

2.1 High-resolution compact implicit scheme

To obtain the high-resolution form of the scheme, we have to control the second order update. To do so, we use two modifications for each numerical flux function. The first one is to define solution-dependent values of ω. The second one is to decrease the value of the second order update using a factor l with values less than one. The resulting parametric form of the numerical flux functions in the high-resolution compact implicit numerical scheme is then presented as follows for the forward step

$$F_{i+1/2}^{+,n+1} = f^+(q_i^{n+1}) - \tfrac{l_i}{2}((1-\omega_i)(f^+(q_i^{n+1}) - f^+(q_{i+1}^{n}))+ \\ \omega_i(f^+(q_{i-1}^{n+1}) - f^+(q_i^{n}))\,,$$

where the parameters $\omega_i \in [0,1]$ and $l_i \in [0,1]$ will be defined later. The values of the parameters are different in the forward and the backward step in the fractional step method for the same grid point x_i. The different values of the parameters are also used for each time step. For the values of the parameters, we compute the special indicator r defined by the following term

$$r_i = \frac{q_{i-1}^{n+1} - q_i^n}{q_i^{n+1} - q_{i+1}^n}\,.$$

Once this special indicator is computed, the limiters like ENO, WENO, TVD, etc., can be used. We prefer to use the high-resolution limiter described in the thesis.

3 System of conservation laws

The system of conservation laws can be written in the vector form as follows

$$\partial_t \mathbf{q} + \partial_x \mathbf{F}(\mathbf{q}) = 0\,, \quad \mathbf{q}(x,0) = \mathbf{q}^0(x)\,, \quad x \in \Omega \subset \mathcal{R}\,, \quad t > 0\,,$$

where $\mathbf{q} = \mathbf{q}(x,t)$ is a vector of unknown functions prescribed at $t = 0$ by the given initial condition $\mathbf{q}^0$, and $\mathbf{F}$=$\mathbf{F}(\mathbf{q})$ is the flux vector function.

The forward step is defined for the system of equations in the following form

$$\mathbf{q}_i^{n+1/2} + \frac{\Delta t}{\Delta x}\mathbf{F}_{i+1/2}^{+,n+1/2} = \mathbf{q}_i^n + \frac{\Delta t}{\Delta x}\mathbf{F}_{i-1/2}^{+,n+1/2}\,,\ i = 1, 2, \ldots, I\,, \quad (5)$$

and the backward step, for the system of equations, has the form

$$\mathbf{q}_i^{n+1} - \frac{\Delta t}{\Delta x}\mathbf{F}_{i-1/2}^{-,n+1} = \mathbf{q}_i^{n+1/2} - \frac{\Delta t}{\Delta x}\mathbf{F}_{i+1/2}^{-,n+1}\,, \;\; i = I-1, I-2, \ldots, 0\,,$$

with the numerical flux functions defined using the characteristic variables. Therefore, in the forward step (5) of the fractional step method, where we consider only $\mathbf{F}^+(\mathbf{q})$, our aim is to express the terms $\mathbf{F}^+(\mathbf{q}_i^{n+1}) - \mathbf{F}^+(\mathbf{q}_{i+1}^n)$ and $\mathbf{F}^+(\mathbf{q}_{i-1}^{n+1}) - \mathbf{F}^+(\mathbf{q}_i^n)$ as the linear combination of the eigenvectors. For the unknown value $\mathbf{q}_i^{n+1}$ in the first term, we use the value $\mathbf{q}_i^{k,n+1}$ from the predictor step or from the previous corrector step. We use auxiliary vector variables $\boldsymbol{\gamma}_i^+ = (\gamma_i^{+,1}, \ldots, \gamma_i^{+,m})$ and $\boldsymbol{\beta}_i^+ = (\beta_i^{+,1}, \ldots, \beta_i^{+,m})$ which are defined as following

$$\boldsymbol{\gamma}_i^+ = (R^+)^{-1} \cdot \left(\mathbf{F}^+(\mathbf{q}_i^{k,n+1}) - \mathbf{F}^+(\mathbf{q}_{i+1}^n)\right)\,,$$

$$\boldsymbol{\beta}_i^+ = (R^+)^{-1} \cdot \left(\mathbf{F}^+(\mathbf{q}_{i-1}^{n+1}) - \mathbf{F}^+(\mathbf{q}_i^n)\right)\,,$$

where the $(R^+)^{-1}$ is the inverse matrix of $R^+(\mathbf{q})$. The flux vector function $\mathbf{F}_{i+1/2}^{+,n+1}$ expressed using the characteristic variables has the form

$$\mathbf{F}_{i+1/2}^{+,n+1} = \mathbf{F}^+(\mathbf{q}_i^{n+1}) - \frac{1}{2}\sum_{p=1}^{m} l_i^p \left((1-\omega_i^p)\gamma_i^{+,p} + \omega_i^p \beta_i^{+,p}\right)\mathbf{r}^{+,p}\,.$$

Note that the parameters $\boldsymbol{\omega}_i$ and $\mathbf{l}_i$ are now associated with the components in $\boldsymbol{\gamma}_i^{\pm}$ and $\boldsymbol{\beta}_i^{\pm}$.

4 System of balance laws

The system of balance laws can be represented in the vector form as follows

$$\partial_t \mathbf{q} + \partial_x \mathbf{F}(\mathbf{q}) = \mathbf{S}(\mathbf{q}) \partial_x H \, , \tag{6}$$

where $\mathbf{q} = \mathbf{q}(x,t)$ is a vector of unknown functions, $\mathbf{F}$=$\mathbf{F}(\mathbf{q})$ is the flux vector function, $\mathbf{S}$=$\mathbf{S}(\mathbf{q})$ is the source vector function and $H = H(x)$ is a given smooth function.

4.1 Well-balanced compact implicit scheme

The numerical scheme uses the fractional step method, and we split the flux vector function into the sum $\mathbf{F} = \mathbf{F}^+ + \mathbf{F}^-$, where the Jacobian has nonnegative and nonpositive eigenvalues for $\mathbf{F}^+(\mathbf{q})$ and $\mathbf{F}^-(\mathbf{q})$ respectively. For the source vector function $\mathbf{S}(\mathbf{q})$, we consider also the splitting $\mathbf{S} = \mathbf{S}^+ + \mathbf{S}^-$ with the simplest choice $\mathbf{S}^\pm = \frac{1}{2}\mathbf{S}$.

The resulting scheme can be written in the form

$$\begin{gathered} \mathbf{q}_i^{n+1} + \tfrac{\Delta t}{\Delta x}\left(\mathbf{F}^{+,n+1}_{i+1/2} - \mathbf{F}^{+,n+1}_{i-1/2}\right) = \\ \mathbf{q}_i^n + \Delta t \left(\tfrac{\mathbf{S}^+(\mathbf{q}_i^{n+1}) + \mathbf{S}^+(\mathbf{q}_i^n)}{2}\right) \partial_x H(x_i) \, , \end{gathered} \tag{7}$$

where the numerical flux vector functions are defined identically as for the systems of conservation laws using the approach of the characteristic variables.

Now, we present the details for the well-balance property. To realize it, one has to solve the following ordinary differential equation (ODE) to find the stationary solution of (6)

$$\partial_x \mathbf{F}(\mathbf{q}^*) = \mathbf{S}(\mathbf{q}^*)\partial_x H(x)\,, \tag{8}$$

with the initial value $\mathbf{q}^*(x_i) = \mathbf{q}_i^n$. The solution of the (8) is the stationary solution $\mathbf{q}_i^{*,n}(x)$. and for the grid nodes we use the following notation $\mathbf{q}_{j,i}^{*,n} := \mathbf{q}_i^{*,n}(x_j)$. The stationary solution $\mathbf{q}_{j,i}^{*,n}$, the so called *local equilibrium*, shall be computed for each grid point x_j for the corresponding stencil $j = i-2,..,i+1$ for the forward step and $j = i-1,..,i+2$ for the backward step.

The procedure for the well-balanced numerical schemes used for our compact implicit scheme is based on the procedure from [26]. The idea is to add the following terms into the scheme (7) for the given stationary solution

$$\frac{\Delta t}{\Delta x}\left(\mathcal{F}_{i+1/2,i}^{+,n} - \mathcal{F}_{i-1/2,i}^{+,n}\right) - \Delta t \mathbf{S}^+(\mathbf{q}_{i,i}^{*,n})H_x(x_i)\,,$$

where

$$\mathcal{F}_{i+1/2,i}^{+,n} = \mathbf{F}^+(\mathbf{q}_{i,i}^{*,n}) - \tfrac{\mathbf{l}_i}{2}((1-\boldsymbol{\omega}_i)(\mathbf{F}^+(\mathbf{q}_{i,i}^{*,n}) - \mathbf{F}^+(\mathbf{q}_{i+1,i}^{*,n}))$$
$$+\boldsymbol{\omega}_i(\mathbf{F}^+(\mathbf{q}_{i-1,i}^{*,n}) - \mathbf{F}^+(\mathbf{q}_{i,i}^{*,n})))\,,$$

and analogously for $\mathcal{F}_{i-1/2,i}^{+,n}$. In the approach of using the characteristic variables the function $\mathcal{F}_{i+1/2,i}^{+,n}$ has the following form

$$\mathcal{F}_{i+1/2,i}^{+,n} = \mathbf{F}^+(\mathbf{q}_{i,i}^{*,n}) - \frac{1}{2}\sum_{p=1}^{m} l_i^p \left((1-\omega_i^p)\gamma_i^{*,+,p} + \omega_i^p \beta_i^{*,+,p}\right)\mathbf{r}^{+,p}\,,$$

where the variables $\boldsymbol{\gamma}_i^{*,+}$ and $\boldsymbol{\beta}_i^{*,+}$ are defined as following

$$\boldsymbol{\gamma}_i^{*,+} = (R^+)^{-1} \cdot \left(\mathbf{F}^+(\mathbf{q}_{i,i}^{*,n}) - \mathbf{F}^+(\mathbf{q}_{i+1,i}^{*,n})\right) ,$$

$$\boldsymbol{\beta}_i^{*,+} = (R^+)^{-1} \cdot \left(\mathbf{F}^+(\mathbf{q}_{i-1,i}^{*,n}) - \mathbf{F}^+(\mathbf{q}_{i,i}^{*,n})\right) .$$

Note that $R^{\pm} = [\mathbf{r}^{\pm,1}, \ldots, \mathbf{r}^{\pm,p}]$ represents the matrix of the eigenvectors of the system (6) for $\mathbf{q} = \mathbf{q}_i^{n+1}$.

The resulting well-balanced high-resolution numerical scheme can be written considering only $\mathbf{F}^+(\mathbf{q})$ and $\mathbf{S}^+(\mathbf{q})$ in the following form

$$\mathbf{q}_i^{n+1} + \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2}^{+,n+1} - \mathcal{F}_{i+1/2,i}^{+,n} - (\mathbf{F}_{i-1/2}^{+,n+1} - \mathcal{F}_{i-1/2,i}^{+,n})\right) =$$
$$\mathbf{q}_i^n + \Delta t\left(\frac{\mathbf{S}^+(\mathbf{q}_i^{n+1}) + \mathbf{S}^+(\mathbf{q}_i^n)}{2} - \mathbf{S}^+(\mathbf{q}_i^{*,n})\right)\partial_x H(x_i) .$$

5 System of balance laws in 2D

In general, the system of balance laws in two dimensions can be written using the following vector form

$$\partial_t \mathbf{q} + \partial_x \mathbf{F}(\mathbf{q}) + \partial_y \mathbf{G}(\mathbf{q}) = \mathbf{S}_1(\mathbf{q})\partial_x H + \mathbf{S}_2(\mathbf{q})\partial_y H ,$$

where $\mathbf{q}(x, y, t)$ is the vector of the unknown conservative variables, $\mathbf{F}(\mathbf{q})$ and $\mathbf{G}(\mathbf{q})$ are the flux vectors in x and y directions respectively, and $\mathbf{S}_1(\mathbf{q})$ and $\mathbf{S}_2(\mathbf{q})$ are the source term vectors. The $H = H(x, y)$ is a given smooth function.

5.1 Well-balanced compact implicit scheme in 2D

The splitting of the flux vectors and the source vectors is considered as $\mathbf{F}(\mathbf{q}) = \mathbf{F}^+(\mathbf{q}) + \mathbf{F}^-(\mathbf{q})$, $\mathbf{G}(\mathbf{q}) = \mathbf{G}^+(\mathbf{q}) + \mathbf{G}^-(\mathbf{q})$ and $\mathbf{S}_{1,2}(\mathbf{q}) = \mathbf{S}^+_{1,2}(\mathbf{q}) + \mathbf{S}^-_{1,2}(\mathbf{q})$. Using the approach of the diagonal sweeps [20], the numerical solution in time step t^{n+1} is obtained by two steps considering, e.g. $\mathbf{F}^+(\mathbf{q})$, $\mathbf{G}^+(\mathbf{q})$, and $\mathbf{S}^+_{1,2}(\mathbf{q})$ in the first step and $\mathbf{F}^-(\mathbf{q})$, $\mathbf{G}^-(\mathbf{q})$, and $\mathbf{S}^-_{1,2}(\mathbf{q})$ in the second step. The well-balanced high-resolution numerical scheme has then the form for the first diagonal step

$$
\begin{gathered}
\mathbf{q}^{n+1/2}_{i,j} + \tfrac{\Delta t}{\Delta x}\left(\mathbf{F}^{+,n+1/2}_{i+1/2,j} - \mathcal{F}^+_{i+1/2,j} - (\mathbf{F}^{+,n+1/2}_{i-1/2,j} - \mathcal{F}^+_{i-1/2,j})\right) \\
+ \tfrac{\Delta t}{\Delta y}\left(\mathbf{G}^{+,n+1/2}_{i,j+1/2} - \mathcal{G}^+_{i,j+1/2} - (\mathbf{G}^{+,n+1/2}_{i,j-1/2} - \mathcal{G}^+_{i,j-1/2})\right) = \\
\mathbf{q}^n_{i,j} + \Delta t\left(\frac{\mathbf{S}^+_1(\mathbf{q}^{n+1/2}_{i,j}) + \mathbf{S}^+_1(\mathbf{q}^n_{i,j})}{2} - \mathbf{S}^+_1(\mathbf{q}^*_{i,j})\right)\partial_x H(x_i, y_j) + \\
\Delta t\left(\frac{\mathbf{S}^+_2(\mathbf{q}^{n+1/2}_{i,j}) + \mathbf{S}^+_2(\mathbf{q}^n_{i,j})}{2} - \mathbf{S}^+_2(\mathbf{q}^*_{i,j})\right)\partial_y H(x_i, y_j)\,, \\
i, j = 1, 2, \ldots, I\,.
\end{gathered}
$$

For the opposite diagonal step, an analogous form is used.

The approach of the characteristic variables is used, and the numerical flux vector function are defined similarly to the one dimensional case.

6 Numerical experiments

For the scalar conservation laws, we solve the Burgers' equation for the different values of the scheme parameters, and we compare the results between different parameters or with the first order implicit numerical scheme or if it is available with the exact solution.

For the system of hyperbolic equations, we solve Euler equations, where the numerical solution consists of several waves with different behaviour. We test the numerical scheme using the characteristic variables approach for the case of nonlinear systems of hyperbolic equations.

To test the well-balanced form of the scheme, we solve the shallow water equations with topography for the system of equations. We test the ability of the scheme to preserve the stationary solutions using them as the initial conditions. Furthermore, we performed the perturbations of the stationary solutions.

Using the well-balanced high-resolution numerical scheme in two dimensions, we test the scheme to preserve the stationary solution of the shallow water equations with topography.

6.1 Burgers' equation

This numerical experiment for the Burgers' equation combines both Riemann's problems, and one observes a shock wave followed by the rarefaction wave [15, 20]. We aim to test the behaviour of the high-resolution scheme in the nontrivial interaction between rarefaction and shock wave for a large time step $\Delta t = 4\Delta t$.

We present the results in Figure 1 in the time $t = 1/2$ where the waves merge and in the final time $T = 1$. The behaviour of the high-resolution scheme is perfect, especially around the merging point of two waves. We obtained a higher accuracy than the first order scheme.

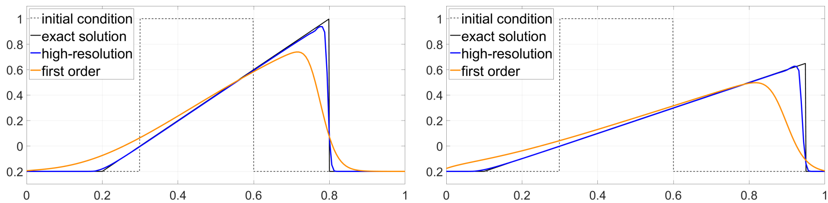

Figure 1: Initial condition and the comparison of the exact and numerical solutions for the example with triangular profile obtained with the first order numerical scheme and the high-resolution scheme with the HR limiter. The results are plotted for the mesh with $I = 160$ and maximal Courant number $C = 4$. The left picture represents the results in the waves merging time $t = 1/2$ and the right picture in the final time $T = 1$

6.2 Euler equations

We solve the numerical experiment called Sod's shock tube problem to test the ability of the scheme to solve the shock wave, rarefaction wave and contact discontinuity for the system of Euler equations. The problem is a special case of the Riemann problem.

The shock tube is filled with the same gas at different pressures and densities, separated by a membrane placed in the centre of the tube. At the initial time, the membrane is ruptured. The solution consists of the shock wave moving into the part at lower pressure and the rarefaction wave moving into the part at higher pressure. The interface between gas states is contact discontinuity. We solved this numerical experiment with two different meshes with $I = 80, 640$, and the Courant number's maximal value is $C = 4.4$. The value of the parameters of the scheme is set using the HR limiter. The solution is compared with the solution obtained by the first order implicit numerical scheme and with the exact solution.

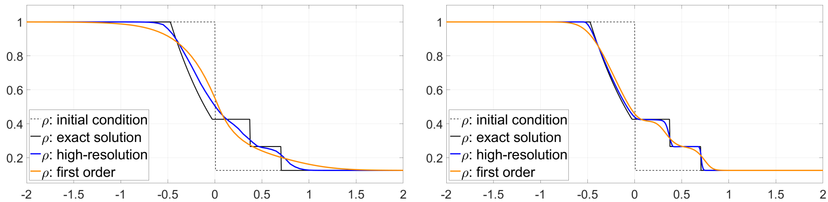

Figure 2: The initial condition and the comparison of the exact and numerical solutions for the density ρ obtained with the first order numerical scheme and the high-resolution scheme with the HR limiter. The results are plotted for the mesh with $I = 80$ in the left picture and the mesh with $I = 640$ in the right picture. The final time is $T = 0.4$, and the maximal Courant number is $C = 4.4$

6.3 Burgers' equation with source term

We first solve the scalar balance laws using the well-balanced high-resolution (WBHR) scheme. The following equation represents the Burgers' equation with the source term

$$\partial_t q + \partial_x(\frac{q^2}{2}) = q^2 \partial_x H \,,$$

where $H = H(x)$ is a given function in the form

$$H(x) = x + 0.1 \sin(100x) \,,$$

and the experiment is computed until the final time $T = 8$. For the perturbation of the stationary solution, we used the initial condition in the following form

$$q(x, 0) = e^{x + 0.1 \sin(100x)} + 0.1 e^{-200(x+0.5)^2} \,.$$

The initial perturbation left the computational domain, and the stationary solution is preserved in the final time T. The results are plotted in Figure 3 for the mesh with $I = 160$.

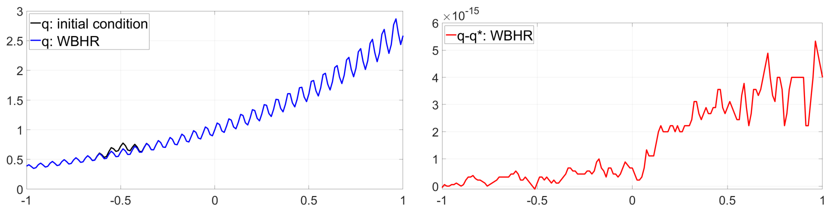

Figure 3: Initial condition and solution obtained by the well-balanced scheme for the mesh with $I = 160$ with the HR limiter in the left figure. Difference between the numerical solution and the stationary solution in the right figure. The final time is $T = 8$ and maximal Courant number is $C = 6.4$

6.4 Shallow water equations with topography in 1D

In the following numerical experiment, we test the ability of the scheme to preserve the moving water equilibria. We use the computational domain $\Omega = [0, 2]$ and the topography $H(x)$ given by the following function

$$H(x) = \begin{cases} 0.25(1 + \cos(10\pi(x - 1.5))) & 1.4 \le x \le 1.6 \\ 0 & \text{otherwise}\,. \end{cases}$$

We perform the perturbations of the stationary solutions at the initial condition as follows

$$h(x,0) = \begin{cases} h^*(x) + \epsilon & 1.1 \le x \le 1.2\,, \\ h^*(x) & \text{otherwise}\,, \end{cases} \qquad u(x,0) = u^*(x)\,. \quad (9)$$

The numerical experiments are computed using several meshes, and the maximal Courant number is always greater than one. The boundary conditions for the examples with the initial condition as the stationary solution and its perturbations are set using the values from the stationary solution.

6.4.1 Subcritical flow

The experiments are performed with the subcritical flow. We test the ability of the scheme to preserve the stationary solution, we perform the perturbation to the initial condition as the stationary solution, and we compare the convergence of the solution in the final time $T = 0.1$. For the visual comparison, the reference solution is computed using the WBHR scheme with the Courant number $C = 0.7$ and using the mesh with $I = 2560$.

The results for the case where the initial condition is set as the stationary solution are presented in Figure 4. We plotted the variables h, u, q and the topography $H(x)$. The difference between the stationary and numerical solution for the variable q is also plotted.

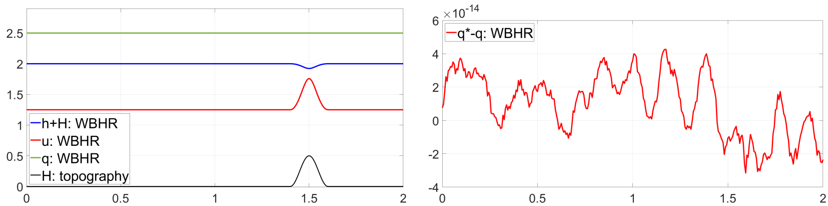

Figure 4: Solution obtained using the WBHR scheme with the mesh with $I = 320$ for the subcritical flow in the left figure. Difference between the stationary and numerical solution for the variable q in the right figure. The final time is $T = 2$ and maximal Courant number is $C = 5.8$

The convergence of the solution with the initial condition set as (9) with the $\epsilon = 0.001$ is shown in Figure 5, where we present the convergence for the solutions for the variable h and u. We compare the solutions to the reference ones.

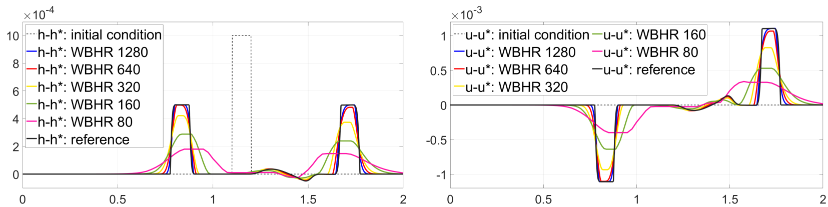

Figure 5: Convergence of the numerical solution for the subcritical flow with the initial perturbation with $\epsilon = 0.001$. The final time is $T = 0.1$ and the maximal Courant number is $C = 5.8$

6.5 Shallow water equations with topography in 2D

The steady vortex numerical experiment [21] is chosen to test the numerical scheme in two dimensions. For two dimensional stationary solution, we test the ability of the scheme to preserve the steady state.

The topography $H(x, y)$ is given by the smooth function

$$H(x, y) = 0.2e^{0.5(1-(x^2+y^2))} .$$

For the initial and boundary conditions, the exact solution is used. The results are plotted in Figure 6, The well-balanced high-resolution scheme preserved the stationary solution using the HR limiter up to the machine accuracy.

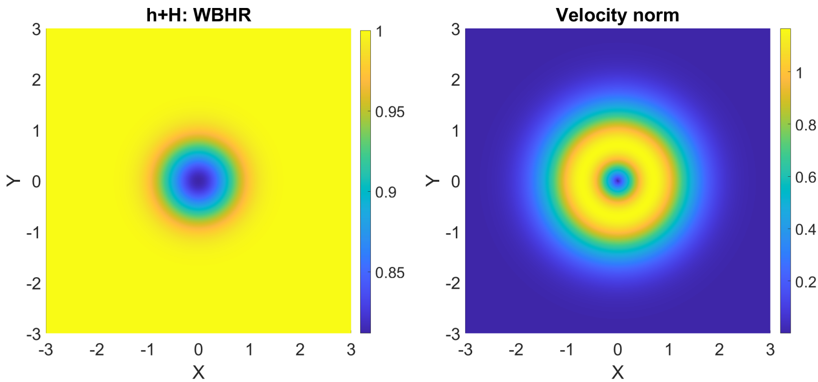

Figure 6: Top view on the numerical solution of the steady vortex obtained by the WBHR scheme for the free surface $h + H$ in the left figure and for the norm of the velocity in the right figure. The solution is computed using HR limiter mesh 128×128 until the final time $T = 1.125$, and the maximal Courant number in each direction is $C = 3.78$

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